Vector Analysis-Outline

1. Definition:
Scalar; symbole, m, example: mass, temperature, potential
vector; symbole, $\overline{a}$, example: force, velocity, displacement

2. Notation-Unit vector-rectangular coordinates
Right-hand system
$\overline{i}$, $\overline{j}$, $\overline{k}$ are unit vectors (also written i, j, k)

$$\overline{a} = ia_x + ja_y + ka_z$$

$$\overline{b} = ib_x + jb_y + kb_z$$

$$\overline{V} = iu_x + jv_y + kw_z$$

3. The dot or Scalar Product

$$\overline{AB} = AB\cos\theta$$

$$\overline{AB} = \overline{BA} \text{ (commutative law)}$$

$$\overline{A}(\overline{B} + \overline{C}) = \overline{AB} + \overline{AC} \text{ (distributive law)}$$

$$m(\overline{AB}) = (m\overline{A})\overline{B} = \overline{A}(m\overline{B}) = (\overline{AB})m$$

$$ii = jj = kk = 1, \quad ij = jk = ki = 0$$

$$if\ \overline{A} = iA_x + jA_y + kA_z, \quad \overline{B} = iB_x + jB_y + kB_z$$

$$C = \overline{AB} = A_xB_x + A_yB_y + A_zB_z, \quad |C| = AB\cos\theta$$

$$\overline{AA} = A^2 = A_x^2 + A_y^2 + A_z^2$$

$$\overline{BB} = B^2 = B_x^2 + B_y^2 + B_z^2$$

4. Cross Product

$$\overline{D} = \overline{A} \times \overline{B} = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}, \quad |D| = AB\sin\theta$$

$$where\ \overline{A} = iA_x + jA_y + kA_z, \quad \overline{B} = iB_x + jB_y + kB_z$$

$$\overline{A} \times \overline{B} = -\overline{B} \times \overline{A} \text{ (commutative law fails)}$$

$$\overline{A} \times (\overline{B} + \overline{C}) = \overline{A} \times \overline{B} + \overline{A} \times \overline{C} \text{ (distributive law)}$$

$$m(\overline{A} \times \overline{B}) = m(\overline{A}) \times \overline{B} = \overline{A}m(\times\overline{B}) = (\overline{A} \times \overline{B})m$$

$$i \times i = j \times j = k \times k = 0,\ i \times j = k,\ j \times k = i,\ k \times i = j$$

5. Differentiation Fofrmulas

$$\frac{d}{du}(\overline{A} + \overline{B}) = \frac{d\overline{A}}{du} + \frac{d\overline{B}}{du}$$

$$\frac{d}{du}(\overline{AB}) = \overline{A}\frac{d\overline{B}}{du} + \frac{d\overline{A}}{du}\overline{B}$$

$$\frac{d}{du}(\overline{A} \times \overline{B}) = \overline{A} \times \frac{d\overline{B}}{du} + \frac{d\overline{A}}{du} \times \overline{B}$$

$$\frac{d}{du}(\phi \times \overline{A}) = \phi\frac{d\overline{B}}{du} + d\frac{\phi}{du}\overline{A}$$

$$\frac{d}{du}(\overline{AB} \times \overline{C}) = \overline{AB} \times \frac{d\overline{C}}{du} + \overline{A}\frac{d\overline{B}}{du} \times \overline{C} + \frac{d\overline{A}}{du}\overline{B} \times \overline{C}$$

$$\frac{d}{du}\{\overline{A} \times (\overline{B} \times \overline{C})\} = \overline{A} \times (\overline{B} \times \frac{d\overline{C}}{du}) + \overline{A} \times (\frac{d\overline{B}}{du} \times \overline{C}) + \frac{d\overline{A}}{du} \times (\overline{B} \times \overline{C})$$

The order in these products may be important.

6. Gradient, Divergence and Curl (Rotation)

The vector differential operator-Del

$$\text{Define}: \ \nabla \equiv \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

(The Gradient)

$$\nabla\phi \ (grad\phi) = (\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k)\phi = (\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k)\phi$$

Note that $\nabla\phi$ defines a vector field.

(The Divergence)

$$Let\ \overline{V} = V_x i + V_y j + V_z k$$

$$\nabla\overline{V} = (\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k)(V_x i + V_y j + V_z k)$$

$$\frac{\partial V_x}{\partial x}i + \frac{\partial V_y}{\partial y}j + \frac{\partial V_z}{\partial z}k$$

Note that $\nabla\overline{V} \neq \overline{V}\nabla$

(The Curl or Rotation)

$$Let\ \overline{V} = V_x i + V_y j + V_z k$$

$$\nabla \times \overline{V} = (\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k) \times (V_x i + V_y j + V_z k)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

Example 1. (Gradiednt)
If $\phi\ (x,\ y,\ z) = 3x^2y - y^3z^2$, find $\nabla\phi$ at the point (1, -2, -1).
(Sol.)

$$\nabla\phi = (\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k)(3x^2y - y^3z^2)$$

$$= 6xyi + (3x^2 - 3y^2z^2)j - 2y^3zk$$

$$At\ point\ (1,\ -2,\ -1),\ \nabla\phi = -12i - 9j - 16k$$

Example 2. (Divergence)
(Sol.) Find $\nabla \phi if \phi = \ln |\overline{r}|$.

$$where\ \overline{r} = xi + yj + zk$$

$$|\overline{r}| = \sqrt{x^2 + y^2 + z^2}, \quad \phi = \ln |\overline{r}| = \frac{1}{2} \ln(x^2 + y2 + z^2)$$

$$\nabla\phi = \frac{1}{2}\{i\frac{2x}{x^2 + y^2 + z^2} + j\frac{2x}{x^2 + y^2 + z^2} + k\frac{2x}{x^2 + y^2 + z^2}\}$$

$$= \frac{xi + yj + zk}{x^2 + y^2 + z^2} = \frac{|\overline{r}|}{r^2}$$

Example 3. (Divergence)
Given $\phi = 2x^3y^2z^4,\ find\ \nabla\nabla\phi\ (div\ grad\phi)$.
(Sol.)

$$\nabla\phi = i\frac{\partial}{\partial x}(2x^3y^2z^4) + j\frac{\partial}{\partial y}(2x^3y^2z^4) + \frac{\partial}{\partial z}(2x^3y^2z^4)$$

$$= 6x^2y^2x^4i + 4x^3yz^4j + 8x^3y^2z^3k$$

$$\nabla\nabla\phi = \frac{\partial}{\partial x}(6x^2y^2z^4) + \frac{\partial}{\partial y}(4x^3yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3)$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

Example 4. (Curl or Rot)
If $\overline{A} = x^2yi - 2xzj + 2yzk$, find curl curl $\overline{A}$.

(Sol.)

$$\nabla \times (\nabla \times \overline{A}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & -2xz & 2yz \end{vmatrix}$$

$$= \nabla \times \{(2x + 2z)i - (x^2 + 2z)k\}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + 2z & 0 & -x^2 - 2z \end{vmatrix} = 2(x + 1)j$$